
IX. *The Conſtruction and Properties of a new* Quadratrix *to the* Hyperbola, *By Mr* . . . Perks. *Communicated by Mr* Abr. de Moivre, *F. R. S.*

THe *Circle*, *Ellipſis* and *Hyberbola* being not Geometrically Quadrable (as infinite others) there have been two ways made uſe of to find their *Area*'s. 1. By *Converging Series*, whereby Approaches are made nearer and nearer, according to the exactneſs deſir'd. 2. By *Quadratices*, that is, Mechanical Curves, which determine the Length of certain Lines, whoſe Squares or Rectangles give the *Area* of the Figure deſir'd. Of this ſort is the old *Quadratrix of Dinoſtratus*, by which the Circle and Ellipſe are ſquared ; and another ſort (for the ſame purpoſe) I inſerted in the *Tranſactions* about 5 years ago. Since that, having found the Conſtruction of a Curve, from whence (beſides its own *Quadrature* and *Rectification*) the *Quadrature* of the *Hyperbola* is deriv'd, I thought the following Account might not (to ſome) be unacceptable.

Let A B, C D, be two ſtraight Rulars joyned at B, and there making a right Angle. (Their length according to the largeneſs of the Figure you will deſcribe.) E F is another Rular ſomewhat longer than A B. Near the one end E, let a little *Truckle-wheel* (repreſented edge-wiſe by *g h*, and made of a thin Plate of Braſs or Iron) be faſtned to the Rular by a Pin (*i*,) thorow its Center, ſo that the Wheel may turn about upon the Pin (*i*) tight to the Rular without joggling.

On the under ſide, of this Rular (the ſide from the Eye in the Scheme) let there be pinn'd or glewed a little piece of Wood (in the form of a Quadrant, the part which is ſeen being mark'd *k l*) whoſe edge (or limb) *k l*, is an arch of a Circle of Center (*i*,) and *Radius i h* (the ſame with the little Wheel.) The deſign of this piece of Wood is, that in the ſeveral Poſitions of the Rular E F, the circular Limb *k l* always touching and ſliding by the edge of the Rular A B, the Center of the Wheel may be always in a line (*i m*) parallel to the Rular A B.

In the Rular C D make M B = *i h* or *i k*, and at M faſten a little Pin, and another to the Rular E F near the Wheel, as at *p*. To theſe two Pins let be faſtned the two ends of a String M R, ſo that its whole length (from Pin to Pin) + *p i*, be equal to the intended Axis of the Curve T W.

The Inſtrument being thus prepar'd, let a ſtrong Rular S O, be faſtned (or held faſt) upon the Paper or Plain that the Curve is to be drawn upon. Lay the Rular E F from M towards A, and parallel to A B, ſo that the String lye all ſtraight along the edge of the Rular E F from M to *p*, the point *S k* of the Quadrantal piece of Wood reſting upon the edge of the Rular A B. Then with a ſmall Pin at M keeping the String cloſe to the edge of the Rular E F, and with your other hand upon the end E, keeping the Wheel tight to the Paper or Plain, move the Pin, String and Rular E F from M towards O, the Rular C D ſliding along by the faſtned Rular S O in a right line, the Wheel *g h* will by its motion deſcribe the deſired Curve T V.

Note, The Semi-diameter of the little Wheel muſt be about the *Sum* of the thickneſſes of the two Rulars E F and A B, that it may touch the Paper. Alſo it will be convenient that its edge be thin, and a little rough, that it may not ſlide flat-ways, and that it may leave a viſible impreſſion.

From this Conſtruction the following Properties are demonſtrable.

I. It is evident from the Conſtruction, that the *Sum of the Tangent and Subtangent* is every where equal to the ſame given Line $= MR + Ri = TW.$) for the String (firſt ſtraight at T W, afterwards making an Angle at R) being every where the ſame, the Line R i (or R p + p i) is always the Tangent, and the Remainder R M the Subtangent; the Contact of the Wheel with the Plain, being the point of the Curve to which they belong.

II. It hence follows, that any aſſignable part of the Curve is *Rectifiable*, or equal to any aſſignable ſtraight Line. In Fig. 2. Let F A E be a part of the Curve, its Vertex F. H D *d* is the Line deſcribed by the motion of the Pin R (in Fig. 1.) and may be ſhewn to be Aſſymptote to the Curve. F H a perpendicular to H D. Let A be given point in the Curve, A D the Tangent, and B D the Subtangent to the ſame point A. Let *a* be another point in the Curve infinitely near to A. to which let *a d* be the Tangent, and *b d* the Subtangent. Draw A G, *a g* perpendicular to F H and A B, *a b* perpendicular to H D. By the Conſtruction $AD + DB = ad + db$. Let a δ be made equal to *a* D, and draw D δ. Then becauſe $ad + bd = AD + DB$. Subtract b D and a D (or a δ) from both Sums (Equals from Equals) there remains $\delta d + dD = Aa + Bb$ (or C a) A a C,

$D d \delta$ are like Triangles (or differing infinitely little from ſuch) therefore $C a\ (B b) : A a :: \delta d : D d$. and compounding $B b + A a : A a :: \delta d + D d : D d$. Alternating $B b + A a : \delta d + D d :: A a : D d$. But $B b + A a = \delta d + D d$ (as is ſhewn above) therefore $A a = D d$. $A a$ is the fluxional Particle of the Curve F A, and $D d$ is the fluxional Particle of the Line H D: Theſe Fluxions or Augments, being equal, and their flowing quantities beginning together, are themſelves therefore equal, *viz.* $F A = H D$.

Let $F G = x$. $G A\ (= H B) = y$. $A D = t$. $B D = S$. So is the Curve $F A = H D = y + S$: that is, *the Curve from the Vertex to any given point therein, is equal to the Sum of its Ordinate, and Subtangent to the ſame point which is its ſecond Property.*

III. The next Property (and whereupon I call it the *Hyperbolic Quadratrix*) is this, In Fig. 2. let F A E be a part of the Curve, (*&c.* as before.) F I K H is a Square upon the line F H. A I L is an Equilater Hyperbola whoſe Vertex is I, its Aſymptotes H O, H R. its Ax H I μ. From a given point L in the Hyperbola (below its Vertex I) draw L A parallel to the Aſymptote R H, interſecting the Diagonal I H in M, F H in G, and touching the Quadratrix in A. I ſay, that the Hyperbolic Area I L M is equal to a Rectangle, whoſe ſides are the *Ordinate* G A, and twice F H, the Ax to the Quadratrix, that is, Trilin. $I L M = 2 F H \times G A$.

Let $F H = a$, $F G = x$, $G A = y$. becauſe of the Hyperbola $G L \times G H\ (L S) = F H q$. therefore $G L = \frac{F H q}{G H}$; and $L M = \frac{F H q}{G H} - G H\ (M G)$ that is,

$$L M = \frac{a a}{a - x} - a + x = \frac{2 a x - x x}{a - x},$$ and conſequently

the fluxion of the Area I L M = [illegible]

In

In the Rectangle triangle A D B, $AB = a - x$, $BD = S$, $AD = t = a - S$; then is $ADq = ABq + BDq$: or $aa\ 2aS + SS = aa - 2ax + xx + SS$, which being reduced, gives $S = \frac{2ax - xx}{2a}$

Let *l a* be a right line ſuppoſed infinitely near and parallel to L A, and interſecting A B in C. Becauſe of like triangles A C a, A B D; A B : B D : : A C : C a that is $a - x : S \left(= \frac{2ax - xx}{2a}\right) : : \dot{x} : \dot{y}$. therefore $\dot{y} = \frac{2ax - xx}{2aa - 2ax}\dot{x}$. Multiply each by 2 a, and 'tis $2a\dot{y}$ $\frac{2ax - xx}{a - x}\dot{x}$. The *Flowing quantity* of $2a\dot{y}$ is $2ay$ and the *flowing quantity* of $\frac{2ax - xx}{a\ \ x}x$ is the Hyperbolic Area I L M (as is ſhewn before.) Theſe two Area's beginning together at F and I, and having every where equal *Fluxions*, or Augments, are therefore themſelves every where equal.

N. The Quadrature of the Trilinear Figure I L M being thus found, any other Area bounded with the Curve line I L. and any other Right Lines is alſo given.

IV. Suppoſing the ſame things as in the precedent Propoſition, I ſay, that the Area of the Quadratrix F a b H F is equal to half the ſquare of F g, wanting the Cube of F g divided by ſix F H, or $FabHF = \frac{xx}{x} - \frac{xxx}{6a}$. The Fluxion of this Area is the Rectangle $CabB = \overline{a - x} \times \dot{y}$ $= \overline{a - x} \times \frac{2ax - xx}{2aa - ax}\dot{x} = x\dot{x} - \frac{xx}{2a}\dot{x}$. The flowing quantity of $x\dot{x}$ is $\frac{1}{2}xx$: And the flowing quantity

tity of $-\frac{xx}{2a}\dot{x}$ is $-\frac{xxx}{6a}$ [as is eaſily ſhewn by bringing back theſe flowing quantities to their reſpective Fluxions.] And hence alſo it follows, that the whole Area continued on infinitely towards F, is *one third of the Square* F I K H; or $\frac{1}{3}$ a a. Eor ſuppoſing x = a the Area above becomes $\frac{aa}{2} - \frac{aa}{6} - \frac{aa}{3}$.

While I was conſidering the other Properties of this Curve, and had given ſome account of them to my Ingenious Friend Mr *John Colſon*, he returned me a Letter with the Addition of the Quadrature of the Curves Area, which I had not then enquired into.

V. Suppoſing ſtill the ſame things, I ſay that the Solid made by the converſion of the Area F a b H F about the Line H b as an Axis, is equal to a Cylinder whoſe Radius is F H = a, and height equal to $\frac{xx}{2a} - \frac{x^3}{2aa} + \frac{x^4}{8a^3}$. And the whole Solid made by converſiou of the whole Figure infinitely continued, is equal to an eighth part of a Cylinder, whoſe Radius and Height are each equal to F H or *a*.

Let $\frac{P}{D}$ expreſs the Proportion of the Periforie and Diameter of a Circle. Then is $\frac{P}{D}$ a b quad. the Area of a Circle whoſe Radius Is a b. And becauſe C a = $\dot{y}$ = $\frac{x - \frac{xx}{2a}\dot{x}}{a -}$ the fluxion of the Solid is $\frac{P}{D}$ *a b.* q. $\times \frac{x - xx}{2a\dot{x}}$

or

or $\frac{P}{D}\overline{a-x}^{2} \times \frac{x - \frac{xx}{2a}\,\dot{x}}{a-x} = \frac{P}{D}\overline{ax - \frac{3}{2}xx + \frac{x^{3}}{2a}}\,\dot{x}$

whoſe flowing quantity is $\frac{P}{D}\,\frac{axx - xxx}{2} + \frac{x^{4}}{8a}$. Which Solid being divided by $\frac{P}{D}aa$ (the Area of a Circle whoſe Radius is a) gives $\frac{xx}{2a} - \frac{xxx}{2aa} + \frac{x^{4}}{8aa}$ for the height of a Cylinder on the ſaid circular Baſe, and equal to the Solid made by converſion of the Area F a b H F about the Line H b as an Axis. When $x = a$ (that is when the whole Figure is turn'd about its Aſymptote) the height $\frac{xx}{2a} - \frac{x^{3}}{2aa} + \frac{x^{4}}{8aa}$ become $\frac{1}{8}a$

VI. The Curve ſurface of the Solid generated by the Converſion of the Figure F a b H F about H B, is equal to the Curve ſurface of a Cylinder, whoſe Radius is a, and height equal to $\frac{x}{2} - \frac{xx}{4a} + \frac{xxx}{12aa}$. And the whole Curve Surface of the Solid infinitely continued, is equal to *one third part of the Curve Surface of a Cylinder whoſe Radius and Height are equal to* F H *or* a. Which may be demonſtrated after the manner of the precedent Propoſition.

VII. The Radius of the Curvature of any Particle of the **Quadratrix** is $\frac{tt}{a-x}$ and this found Geometrically. In Fig. 3; F A E is the *Quadratrix*, H D the Aſymptote, A D the Tangent, B D the Subtangent to a given point A. Make B V = A D. Upon V riſe the perpendicular V W. from A draw A W perpendicular to the Tangent A D, till

it

it meet A W in W. So is A W the Radius of the Curvature at A.

VIII. This Curve may be continued on infinitely above the point F (but by a different and more operoſe way of Conſtruction) whoſe Properties will be theſe. 1. The *Difference* of its Tangent and Subtangent (taking the Subtangent in the Line H S) will be always equal to the ſame given Line F H or *a*. That is, as $t + s = a$, below F, ſo $t - s = a$ above F. 2. As below F the Curve Line is equal to the *Sum* of its Ordinate and Subtangent, ſo above, it is equal to their *Difference* or $-S-y$. 3. As below F, $2ay = ILM$, ſo above $2ay = I\lambda\mu$. All which (and its other Properties) may be demonſtrated as the Precedent *mutatis mutandis*.

IX. With a little variation in the precedent Conſtruction may the *Logarithmick Curve* be conſtructed, which is alſo a *Quadratrix* to the Hyperbola. In Fig. 1. omitting the String M R P, let the diſtance M R be equal to the *Subtangent* of the intended Logarithmick Curve (which, as 'tis known, is invariable.) Stick a Pin at R in the Rular C D, to which apply the Rular E F, ſo that the edge of the little Quadrant *k l*, reſting upon the Rular A B, the diſtance M i be equal to M R. Then keeping the Rular E F tight to the Pin R and Rular A B, ſlide the Rular C D along in a ſtraight Line (by the Rular or Line S O.) So will the Wheel *g h* deſcribe a part of the Logarithmick Curve T V, whoſe *Subtangent* is every where M R.

X. Fig. 2. Let F A E repreſent the *Logarithmick Curve*, whoſe Subtangent is equal to F H. L I λ is an Equilater Hyperbola (*&c.* as before § III.) Let $FG = x$, $Ga = y$. $FH\ (= BD) = a$. $GH\ (= LS) = a - x$. $AC = x$ $Ca = y$. Then $AC : Ca :: AB : BD$. that is $x : y :: a - x$

$x : a :: a : \frac{aa}{a-x}$. therefore $a\dot{y} = \frac{aa}{a-x}\dot{x}$. The *Flowing quantity* of $a\dot{y}$ is ay; and the *Flowing quantity* of $\frac{aa}{a-x}\dot{x}$ is the Hyperbolick Area F I L G (for by the nature of the Hyperbola $GL = \frac{aa}{a-x}$) therefore is the Hyperbolick Area F I L G equal to ay, a Rectangle whoſe ſides are the Subtangent (B D = F H) and Ordinate G A (as here accounted) of the Logarithmick Curve.

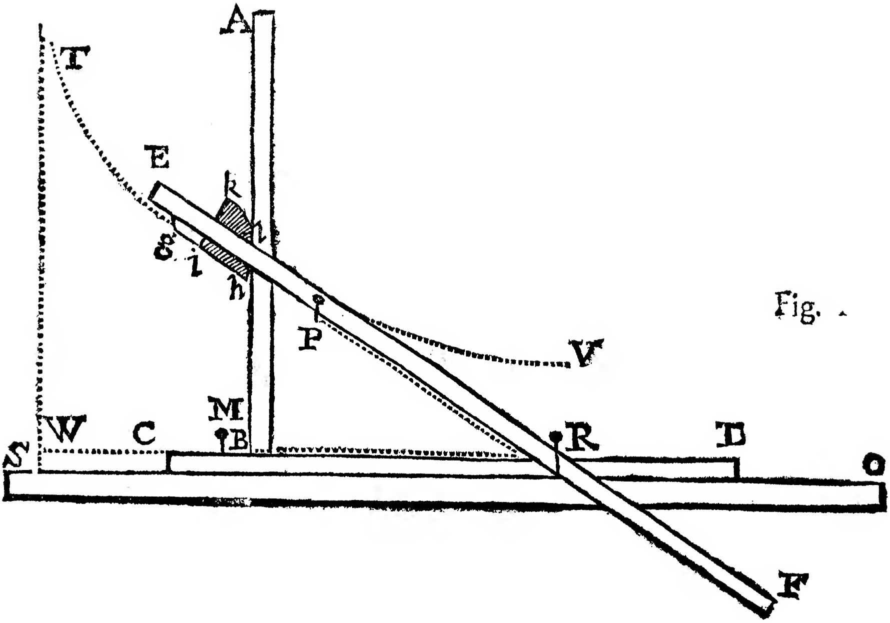

Fig. .

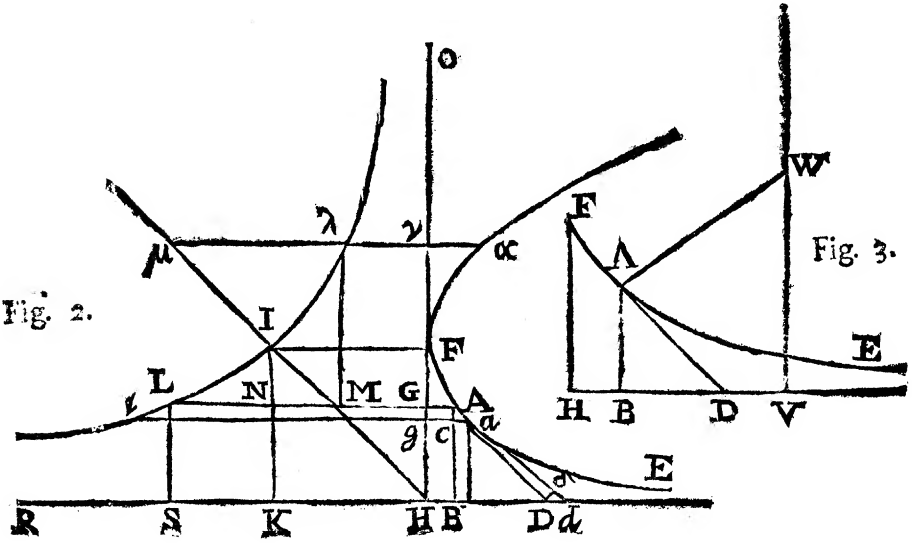

Fig. 2.

Fig. 3.